

# VERTEX ALGEBRAS AND HODGE STRUCTURES

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**ABSTRACT.** In this short note we discuss some natural inter-relations between Hodge structures and vertex algebras of conformal field theory. Some part of this on a correspondence between Higgs bundles and opers already is known in the literature as non-abelian Hodge theorem due to C. Simpson. The same kind of correspondence has been well studied over flag manifolds of semisimple Lie groups known as Beilinson-Bernstein localization. Our goal is to explain how the data of a variation of Hodge structure as a solution of a regular holonomic system is matched with similar system of vertex algebra modules arising in conformal field theory. The result of the discussion is an analogue of the Bernstein correspondence over a local manifold. We associate to flat connections of mixed Hodge structures a generalized version of Harish-Chandra modules called Wakimoto modules and a generalized Harish-Chandra homomorphism. Therefore the map of correspondence is a more developed form of Harish-Chandra isomorphism. This text mainly proposes to motivates some ideas of representations of vertex algebras into Hodge theory. We have brought the basic ideas in the two fields close to each other. We enclose with an explanation of geometric Langlands correspondence as a generalization of the discussion.

## 1. INTRODUCTION

We purpose to compare the context of vertex operator algebras (VOA) of conformal field theory (CFT) with that of mixed Hodge modules (MHM) that arise from Hodge theory or variation of mixed Hodge structures (VMHS). Both of these concepts can be studied from various view of points interested to different areas in mathematics. This note is mostly prepared from a Hodge theoretic view point and is too brief. Vertex algebras naturally arise from highest weight representations of affine or Virasoro algebras. For a basic set up lets consider

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*Key words and phrases.* Vertex operator algebras, Hodge structure, Highest weight modules,  $D$ -modules, KZ-equation, Affine Lie algebras, Root systems, Geometric Langlands correspondence, Wakimoto modules,  $\mathfrak{g}$ -oper, Virasoro algebra, Fock modules.

$$(1) \quad D = t \frac{d}{dt}$$

acting as a differential operator on the ring  $R = \mathbb{C}[t, t^{-1}]$ . The Lie algebra  $\mathfrak{g}$  of derivations of  $R$  is generated by  $t^m D$ ,  $m \in \mathbb{Z}$ . This algebra is graded, where  $t^m D$  has weight  $m$ . We are interested to central extensions

$$(2) \quad 0 \rightarrow \mathbb{C}.c \rightarrow V \rightarrow \mathfrak{g} \rightarrow 0$$

namely the Virasoro algebra. It is also a graded Lie algebra. Certain highest weight representations of  $V$  arise in conformal field theory. They have the property that for a suitable choice of  $\hat{D}$  lifting  $D$  in the extension, the character

$$(3) \quad \text{Trace}(q^{\hat{D}})$$

is well defined and is the  $q$ -expansion of a modular form, [BO]. The definition of Virasoro algebra can be extended to that of vertex algebra. We may think of the Virasoro algebra  $V$  as a vector space acted on by the commuting operators  $v_n$  and  $V$  is generated by  $1 \in V$  and the action of these operators. Then  $V$  becomes a commutative ring such 1 is the identity and the actions of all the operators are given by multiplication of elements of  $V$ . We define an operator

$$(4) \quad \phi(t) : v \mapsto \sum_i D^i v \cdot t^i / i!, \quad V \rightarrow \text{End}(V)[t, t^{-1}]$$

called vertex operator. The maps

$$(5) \quad \text{Trace}(\phi(x)\phi(y)\dots)$$

are called correlation functions from their analogues in quantum field theory. A vertex operator algebra structure is expected to explain a conformal infinitesimal deformation of  $V$ .

A vertex operator algebra is given by a 4-tuple  $(V, Y, 1, \omega)$  where  $V$  is a  $\mathbb{Z}$ -graded vector space with a linear map

$$(6) \quad Y(., z) : V \rightarrow \text{End}(V)[[z, z^{-1}]], \quad Y(v, z) = \sum_{n \in \mathbb{Z}} v_n z^{-n-1}$$

where  $1 \in V_0$  is called the vacuum vector with  $Y(1, z) = id_V$ ,  $v_{-1}1 = v$ . The vector  $\omega \in V_2$  is a specific element called conformal element or Virasoro element such that

$$(7) \quad Y(\omega, z) = \sum_n L(n) z^{-n-1}$$

provides a set of Virasoro generators  $L(n)$  such that  $L(0) |_{V_n} = n \cdot \text{id}_{V_n}$ . The operator  $L(-1)$  satisfies

$$(8) \quad [L(-1), Y(v, z)] = \frac{d}{dz} Y(v, z)$$

and we also assume the Jacobi identity for the vertex operator  $Y$ , cf. [M]. We will consider vertex algebras of CFT-type, that is  $V_n = 0$ ,  $n < 0$  and  $V_0 = \mathbb{C} \cdot 1$ . If  $V$  is generated by the subset  $S \subset V$ , then

$$(9) \quad V = \text{span}\{v_{n_1}^1 \dots v_{n_k}^k \cdot 1 \mid v_i \in S\}$$

A unitary vertex operator algebra is one with a positive definite Hermitian form. This notion can also be defined for the modules over these algebras via an anti-involution. An anti-linear automorphism is a linear map such that

$$(10) \quad \phi : V \rightarrow V, \quad \phi(1) = 1, \quad \phi(\omega) = \omega, \quad \phi(u_n \cdot v) = \phi(u)_n \phi(v) \quad \forall u, v \in V$$

In contrast to Lie algebras the definition of invariant bilinear form on a vertex algebra is highly complicated. For instance, the contragredient  $(V', Y')$  of a vertex algebra module  $(V, Y)$  is defined via the form

$$(11) \quad \langle Y'(v, x)w', w \rangle = \langle w', Y(e^{xL(1)}(-x^2)^{L(0)}v, x^{-1})w \rangle, \quad v, w \in V, \quad w' \in V'$$

This form should also explain the invariant bilinear form on our vertex algebra. A unitary vertex algebra is one which is equipped with a positive definite hermitian form. For a unitary vertex operator algebra the positive definite Hermitian form

$$(12) \quad (\cdot, \cdot)_{\text{unitary}} : V \times V \rightarrow \mathbb{C}, \quad \exists \lambda \in \mathbb{C}; \quad (u, v) = \lambda(1, 1)$$

is uniquely determined by its value at  $(1, 1)$ . This can be easily proved using the axioms of a vertex algebra and the invariance property in (11).

Vertex algebras and vertex homomorphisms build up a tensor category. This simply means that we can tensor a finite number of vertex algebras

$$(13) \quad \bigotimes_{i=1}^p (V_i, Y_{V_i}, I_i, w_i), \quad w = w_1 \otimes 1 \dots \otimes 1 + \dots + 1 \otimes \dots \otimes 1 \otimes w_p$$

The data of a vertex algebra should satisfy a locality axiom. Unfortunately the detailed description of this concept is out of the scope of this short note. We refer the reader to [FB] for a complete explanation. It briefly means as follows. For any  $A, B \in V$  the two formal power series in two variables obtained by composing  $Y(A, z)$  and  $Y(B, w)$  in two possible ways are equal, possibly after multiplying them by a large power of  $(z - w)$ . It can be stated as

$$(14) \quad (z - w)^N [Y(A, z), Y(B, w)] = 0, \quad \text{some } N \in \mathbb{Z}_+$$

Regarding this we define a normally ordered product as

$$(15) \quad : A(z)B(w) : = \sum_m \left\{ \sum_{m < 0} A_m B_n z^{-m-1} + \sum_{m \geq 0} A_m B_n z^{-m-1} \right\} w^{-n-1}$$

of vertex operators. The product can be inductively extended for more than two factors.

**Remark 1.1.** [B] *Vertex algebras when the operators  $V(a, x)$  are holomorphic are commutative rings with derivations. The notation  $V(u, z)v$  is a deformation of the one  $u^z.v$ . If we have a commutative algebra with a derivation  $D$ , then we can define*

$$(16) \quad V(a, x)b = \sum_{i \geq 0} (D^i a) b x^i / i!$$

*Conversely if  $V$  is a vertex algebra, we can define  $ab = V(a, 0)b$ , and*

$$(17) \quad Da = \text{coefficient of } x^1 \text{ in } V(a, x)b$$

*Then in the new notation we put  $a^x = \sum_i x^i D^i a / i!$ , then*

$$(18) \quad a^x b = \sum_i x^i D^i ab / i!$$

*Here  $x$  is thought to be an element in the formal group  $\hat{G}_a$ . This formal group has its formal group ring, the algebra of polynomials  $H = \mathbb{C}[D]$  and its coordinate ring is the ring of formal power series  $\mathbb{C}[[x]]$ . The tensor category of modules with a derivation is the same as the category of modules over the formal group ring  $H$ . So holomorphic vertex algebras are the same as commutative ring objects in this category. In the non-holomorphic case the expressions  $a^x b^y \dots$ , ( $a, b \in V$ ) are no longer holomorphic and can have singularities. In the new notation the identities of vertex algebra theory are easier to understand; for example*

$$(19) \quad V(a, x)b = e^{xL-1}(V(b, -x)a) \quad \Leftrightarrow \quad a^x b = (b^{x^{-1}} a)^x$$

An intertwining operator between 3 modules  $(W_1, Y_1)$ ,  $(W_2, Y_2)$  and  $(W_3, Y_3)$  is a linear map

$$(20) \quad I(., z) : W_2 \rightarrow \text{Hom}(W_3, W_1)\{z\}, \quad u \mapsto I(u, z) = \sum_{n \in \mathbb{Q}} u_n z^{-n-1}$$

satisfying certain conditions of compatibility. The vertex operator  $Y_M(., z)$  will become an intertwining operator where  $W_3 = W_1 = M$ . One way to define these operators is

$$(21) \quad I(w, z)v = e^{zL(-1)}Y_M(v, -z)w$$

The intertwining operators are important tools in order to define a product structure in the category of vertex algebras. This is because the usual tensor product of two Lie algebra modules is not generally a module. This makes the definition of a product structure satisfying associativity in the tensor category of VOA's considerably complicated. Intertwining operators also explain the base of the theory of conformal blocks. Using the  $L(-1)$ -property mentioned above and intertwining operators one can extract a system of differential equations whose solution systems are these modules. These are  $D$ -modules which certain differential operators act on them, [Y].

A polarized Hodge structure on a  $\mathbb{Q}$ -vector space  $V$ , is given by a representation

$$(22) \quad \phi : \mathbb{U}(\mathbb{R}) \rightarrow \text{Aut}(V_{\mathbb{R}}, \mathbb{Q}), \quad \mathbb{U}(\mathbb{R}) = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}, \quad a^2 + b^2 = 1$$

The group  $G_{\mathbb{R}} = \text{Aut}(V_{\mathbb{R}}, \mathbb{Q})$  is a real simple Lie group. The period domain  $D$  associated to the Hodge structure  $\phi$  is the moduli space of polarized Hodge structures on a fixed vector space  $V$  with the same Hodge numbers. The group  $G_{\mathbb{R}}$  acts transitively on the period domain  $D$  by conjugation;

$$(23) \quad D = \{\phi : S^1 \rightarrow G_{\mathbb{R}} ; \phi = g^{-1}\phi_0 g\}$$

The isotropy group  $H$  of a reference polarized Hodge structure  $(V, Q, \phi)$  is a compact subgroup of  $G_{\mathbb{R}}$ , which contains a compact maximal torus  $T$ . The Lie algebra  $\mathfrak{g}$  of the simple Lie group  $G_{\mathbb{C}}$  is a  $\mathbb{Q}$ -linear subspace of  $\text{End}(V)$ , and the form  $Q$  induces on  $\mathfrak{g}$  a non-degenerate symmetric bilinear form  $B : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{C}$  which upto scale is just the Cartan-Killing form  $\text{tr}(\text{ad}(x)\text{ad}(y))$ . For each point  $\phi \in D$

$$(24) \quad \text{Ad}\phi : \mathbb{U}(\mathbb{R}) \rightarrow \text{Aut}(\mathfrak{g}_{\mathbb{R}}, B)$$

is a Hodge structure of weight 0 on  $\mathfrak{g}$ . This Hodge structure is polarized by  $B$ . Associated to each nilpotent transformation  $N \in \mathfrak{g}$  one defines a limit mixed Hodge structure. The local system  $\mathfrak{g} \rightarrow \Delta^*$  is then equipped with the monodromy  $T = e^{\text{ad} N}$  and Hodge filtration defined with respect to the multi-valued basis of  $\mathfrak{g}$  by  $e^{\log(t)\frac{N}{2\pi i}} F^\bullet$ , where  $F^\bullet$  is the natural Hodge filtration on  $\mathfrak{g}$  by (20). It gives a limit MHS  $(\mathfrak{g}, F^\bullet, W(N)_\bullet)$ . The polarizing form gives perfect pairings

$$(25) \quad B_k : Gr_k^{W(N)} \mathfrak{g} \times Gr_{-k}^{W(N)} \mathfrak{g} \rightarrow \mathbb{Q}, \quad B_k(u, v) = B(v, N^k v)$$

defined via the hard Lefschetz isomorphism  $N^k : Gr_{-k}^{W(N)} \mathfrak{g} \cong Gr_k^{W(N)} \mathfrak{g}$ .

A family of projective manifolds defined via a proper smooth map

$$(26) \quad f : X \rightarrow S, \quad X_s = f^{-1}(s)$$

of quasi-projective varieties there associates a polarized variation of Hodge structures (VHS)

$$(27) \quad (\mathcal{V} = R^k f_* \mathbb{C}, F^\bullet)$$

If  $\dim S = 1$  this variation simply is understood as a topological deformation of the Hodge structure  $V = H^k(X_s, \mathbb{C})$  over a punctured disc. The study of the asymptotic behavior of the VHS is an important issue in Hodge theory. We will always assume  $V$  is equipped with the limit Hodge filtration. By the Riemann-Hilbert correspondence a local system of Hodge structures defines a  $D$ -module with flat connection on the base manifold  $S$ . This gives rise to a decreasing filtration denoted also by  $F^\bullet = (F^i)$  on the vector bundle  $\mathcal{V} \otimes_{\mathbb{Q}} \mathcal{O}_S$  by holomorphic sub-bundles, and a flat connection

$$(28) \quad \nabla : \mathcal{V} \otimes_{\mathbb{Q}} \mathcal{O}_S \rightarrow \mathcal{V} \otimes_{\mathbb{Q}} \Omega_S^1$$

satisfying Griffiths transversality;

$$(29) \quad \nabla(F^i \mathcal{V}) \subset F^{i-1} \mathcal{V} \otimes \Omega_S^1$$

These data are also equipped with a flat bilinear pairing

$$(30) \quad P : \mathcal{V} \times \mathcal{V} \rightarrow \mathbb{Q}$$

We will refer to these  $D$ -modules as (polarized) mixed Hodge modules (MHM). In this text we attempt to correspond MHM's as filtered flat connections to vertex algebras in the context explained above. Both of these structures arise from complex system of holomorphic differential equations on complex manifolds with regular singularities.

**Explanation on the text:** Section 1 is the introduction and we introduce the concept from the literature. First, the notion of vertex algebra is introduced and second, the notion of Hodge structure. The basic properties of vertex algebra, intertwining operators and locality axiom are presented to give a basic insight.

In Section 2 we present main examples of vertex algebras we are dealing with, as affine Kac-Moody algebras and Virasoro algebras and Fock modules. We present Fock representations of Heisenberg algebra and Harish-Chandra pairs in this section. A brief description of conformality and unitary representations of VOA's is also given.

In Section 3 we give basic concepts related to variation of (mixed) Hodge structure. We explain the context of mixed Hodge modules and the non-abelian Hodge theorem of C. Simpson as equivalent notions. The purpose of this section is to give different insights toward VHS which classically are related to conformal field theory.

Section 4 contains the Beilinson-Bernstein localization functor which we successively develop over the  $\mathfrak{g}$ -opers in order to explain the geometric Langlands correspondence. We give a brief explanation of KZ-equations and the conformal blocks at the end.

## 2. VERTEX ALGEBRAS

In this section main examples of vertex algebras and their representations are presented along what we explained in the introduction.

**Definition 2.1.** *A vertex algebra consists of the following data;*

- (space of states) *A  $\mathbb{Z}$ -graded vector space*

$$(31) \quad V = \bigoplus_n V_n$$

- (vacuum vector) *a vector  $|0\rangle \in V_0$*
- (translation operator) *a linear operator  $T : V \rightarrow V$  of degree one.*
- (vertex operators) *a linear operation*

$$(32) \quad Y(., z) : V \rightarrow \text{End}(V)[[z^{\pm 1}]]$$

*taking  $A \in V_m$  to*

$$(33) \quad Y(A, z) = \sum_n A_{(n)} z^{-n-1}$$

*of conformal dimension  $m$ , i.e  $\deg A(n) = -n + m = 1$ .*

- (vacuum axiom)  *$Y(|0\rangle) = \text{Id}_V$ . Furthermore*

$$(34) \quad Y(A, z)|0\rangle \in V[[z]], \quad \forall A \in V$$

- (translation axiom) For any  $A \in V$ ,

$$(35) \quad [T, Y(A, z)] = \partial_z Y(A, z)$$

and  $T|0\rangle = 0$ .

- (locality axiom) All fields are local with respect to each others.

Vertex algebra structure naturally appear in known geometric concepts we may know. Lets begin with Lie algebra  $H$  defined as central extension

$$(36) \quad 0 \rightarrow \mathbb{C}.1 \rightarrow H \rightarrow \mathbb{C}((t)) \rightarrow 0$$

It may also be regarded as the completion of the one dimensional central extension of the commutative Lie algebra of Laurent polynomials  $\mathbb{C}[t, t^{-1}]$  having basis  $b_n = t^n$ ,  $n \in \mathbb{Z}$  and the central element 1. Lets call the latter Lie algebra by  $H'$ . The universal enveloping algebra  $U(H')$  is an associative algebra with generators  $b_n$  and relations

$$(37) \quad b_n b_m - b_m b_n = n \delta_{n, -m} 1, \quad b_n \cdot 1 - 1 \cdot b_n = 0$$

The left ideals  $t^N \mathbb{C}[t]$  build up a system of open neighborhoods of 0, and one can consider the completion of  $U(H')$  with respect to this topology, denoted  $\tilde{U}(H')$ . The quotient

$$(38) \quad \tilde{H} = \tilde{U}(H') / (1 - 1)$$

is the well known Weyl algebra. Here the first 1 is the central element and the second is the unit of  $\tilde{U}(H')$ . Let  $\tilde{H}_+$  be the subalgebra of  $\tilde{H}$  generated by  $b_n$ ,  $n \geq 0$  and define

$$(39) \quad V = \text{Ind}_{\tilde{H}_+}^{\tilde{H}} \mathbb{C} = \tilde{H}_- = \mathbb{C}[b_{-1}, b_{-2}, \dots]$$

The module  $V$  is called the Fock representation of  $\tilde{H}$ . Now lets look at to the fields

$$(40) \quad b(z) = \sum_n b_n z^{-n-1}$$

where  $b_n$  is considered as an endomorphism of  $V$ . Since  $\deg(b_n) = -n$ ,  $b(z)$  is a field of conformal dimension one. Lets consider

$$(41) \quad b(z)^2 = \sum_n \left( \sum_{k+l=n} b_k b_l \right) z^{-n-2}$$



The relations (37) imply that the coefficient operators can be rearranged so that the annihilation operators ( $b_n$ ,  $n < 0$ ) be in the right side of creation operators ( $b_n$ ,  $n \geq 0$ ) and for any  $x \in V$  there are only a finite number of  $b_k b_l$  whose action on  $x$  is non zero. This makes the expression (41) well defined. There are standard ways in conformal field theory to remove infinite sums arising from repeatedly creating and annihilating the same state. In our case we define the normally ordered product of  $b(z)$  with itself as

$$(42) \quad : b(z)b(z) = \sum_n : b_k b_l : z^{-n-2}, \quad : b_k b_l := \begin{cases} b_l b_k & l = -k, k \geq 0 \\ b_k b_l & \text{otherwise} \end{cases}$$

With the normally ordered product we can proceed to define for instance

$$(43) \quad Y(b_{-1}^2, z) = : b(z)^2 :$$

etc .... The pattern explained above appears in many Lie algebra representations in finite or infinite dimensions. We will encounter several examples of this in the following.

- (1) **Affine Kac-Moody algebras:** The first series of vertex operator algebras are affine Lie algebras. Let  $\mathfrak{g}$  is a simple Lie algebra of finite dimensional over  $\mathbb{C}$ . Let  $L\mathfrak{g}$  be the loop algebra  $\mathfrak{g}((t))$ . As a vector space an affine algebra is of the form  $\widehat{\mathfrak{g}} := L\mathfrak{g} \oplus \mathbb{C}.K$ , with commutation relations  $[K, \cdot] = 0$  and

$$(44) \quad [A \otimes f(t), B \otimes g(t)] = [A, B] \otimes f(t)g(t) + (Res_{t=0} f dg(A, B)).K$$

where  $(\cdot, \cdot)$  is an invariant bilinear form on  $\mathfrak{g}$ , normalized such that  $(\theta, \theta) = 2$  where  $\theta$  is the highest root of  $\mathfrak{g}$ .

A related concept is the vacuum representation of an affine algebra  $\widehat{\mathfrak{g}}$ . Let  $k \in \mathbb{C}$ , and suppose  $\mathbb{C}_k$  be the 1-dimensional representation of  $\mathfrak{g}[[t]] \oplus \mathbb{C}.k$ . The vacuum representation of  $\widehat{\mathfrak{g}}$  of level  $k$  is

$$(45) \quad V_k(\mathfrak{g}) = Ind_{\mathfrak{g}[[t]] \oplus \mathbb{C}.K}^{\widehat{\mathfrak{g}}} \mathbb{C}_k$$

If  $J^a$ 's be a basis of  $\mathfrak{g}$ , then  $J_n^a = J^a \otimes t^n$  and  $K$  form a basis of  $\widehat{\mathfrak{g}}$  and  $V_k(\mathfrak{g})$  is generated by the monomials  $J_{n_1}^{a_1} \dots J_{n_m}^{a_m} 1_k$ . We obtain a vertex algebra (module)  $V_k(\mathfrak{g})$  with vertex operator

$$(46) \quad Y(J_{-1}^a \cdot 1_k, z) = J^a(z) := \sum_n J_n^a z^{-n-1}$$

The module  $V_k(\mathfrak{g})$  has a unique maximal proper  $\widehat{\mathfrak{g}}$ -submodule  $J(k)$  and

$$(47) \quad L_{\mathfrak{g}}(k, 0) = V_k(\mathfrak{g})/J(k)$$

becomes a simple vertex algebra. We denote by  $L_{\mathfrak{g}}(k, \lambda)$  the corresponding highest weight module for  $\widehat{\mathfrak{g}}$  associated to the highest weight  $\lambda \in \mathfrak{h}^*$  of  $\mathfrak{g}$ , [DX], [F1].

- (2) **Virasoro algebras:** The second type of vertex algebras we consider are Virasoro algebras denoted  $\text{Vir}$ . It is a central extension of the lie algebra  $\text{Der}\mathbb{C}((t))$  generated by the operators  $L_n = -t^{n+1}d/dt$ ,  $n \in \mathbb{Z}$  by the 1-dimensional vector space  $\mathbb{C}C$  with relations,  $[C, \cdot] = 0$  and

$$(48) \quad [L_n, L_m] = (n - m)L_{n+m} + \frac{n^3 - n}{12}\delta_{n, -m}C$$

Assume  $c, h \in \mathbb{C}$  and Let  $\mathbb{C}_c$  be the one dimensional representation of  $\text{Vir}$  defined by

$$(49) \quad \begin{aligned} L_n.1 &= 0 & n \geq 1 \\ L_0.1 &= h.1 \\ c.1 &= c.1 \end{aligned}$$

Define

$$(50) \quad V(c, h) := \text{Ind}_{\text{Der}\mathbb{C}[[t]] \oplus C}^{\text{Vir}} \mathbb{C}_c, \quad c \in \mathbb{C}$$

Then  $V(c, h)$  is generated by the monomials  $L_{j_1} \dots L_{j_m} 1_c$ ,  $j_1 \leq j_2 \leq \dots \leq -2$  and it is a vertex algebra module with vertex operator

$$(51) \quad Y(L_{-2}1_c, z) = T(z) = \sum_n L_n z^{-n-2}$$

The modules over the Virasoro algebra are classified according to the action of the operator  $L(0)$  by  $L(0).1 = h.1$  and it becomes a highest weight module of the Virasoro algebra denoted  $V(c, h)$ . Its unique irreducible quotient is denoted by  $L(c, h)$ , [DX], [IK].

- (3) **(Bosonic) Fock representations:** Fock modules are defined using Heisenberg lie algebra of rank 1.

$$(52) \quad H = \bigoplus_n \mathbb{C}a_n \oplus \mathbb{C}K, \quad [K, \cdot] = 0, \quad [a_m, a_n] = m\delta_{m+n, 0}K$$

Let  $\mathbb{C}_\eta$  be the 1-dimensional  $H^\geq = (H^0 \oplus H^+)$ -representation with

$$(53) \quad a_n 1_\eta = \eta \delta_{n,0} 1_\eta, \quad K.1_\eta = 1_\eta$$

It has a  $\mathbb{Z}$ -gradation with

$$(54) \quad \mathbb{C}_\eta^n = \begin{cases} \mathbb{C}_\eta & n = 0 \\ 0 & n \neq 0 \end{cases}$$

The corresponding (bosonic) Fock module is defined by

$$(55) \quad \mathcal{F}^\eta = \text{Ind}_{H \geq}^H \mathbb{C}_\eta$$

The highest weight vector  $1_\eta$  is denoted  $|\eta\rangle$ . We define a  $\mathbb{Z}$ -graded vertex algebra structure on  $\mathcal{F}^0$  with vacuum vector  $|0\rangle$ , translation operator

$$(56) \quad T|0\rangle = 0, \quad [T, a_n] = -na_{-n-1}$$

and vertex operator,

$$(57) \quad Y(a_{-1}|0\rangle, z) = \sum_n a_n z^{-n-1}$$

Fock representations can be understood as the smallest representations of the Weyl algebra, [IK], [F1].

**(4) Harish-Chandra modules:** A pair  $(\mathfrak{g}, K)$  where  $\mathfrak{g}$  is a Lie algebra,  $K$  is a Lie group, such that  $\mathfrak{k} = \text{Lie}(K)$  and an action

$$(58) \quad \text{Ad} : K \rightarrow \mathfrak{g}$$

compatible with the adjoint action of  $K$  on  $\mathfrak{k}$  is called a Harish-Chandra pair. A  $(\mathfrak{g}, K)$ -action on a scheme  $X$  is a homomorphism

$$(59) \quad \rho : \mathfrak{g} \rightarrow \Theta_X$$

together with an action of  $K$  on  $X$  such that

- (1) the differential of the  $K$ -action is the restriction of the action of  $\mathfrak{g}$  on  $\mathfrak{k}$ .
- (2)  $\rho(\text{Ad}(k)(a)) = k\rho(a)k^{-1}$ .

A (Harish-Chandra)  $(\mathfrak{g}, K)$ -module is a vector space  $V$  with the aforementioned compatible actions. One can consider the vector bundle

$$(60) \quad \mathcal{V} = X \times_K V$$

on the scheme  $X$ , which gives a flat connection on the trivial vector bundle  $X \times V$  on  $X$ .

A Harish-Chandra module  $M$  can also be defined over Virasoro algebras. Via this generalization the Lie algebra  $\mathfrak{g}$  has a generalized triangle decomposition

$$(61) \quad \mathfrak{g} = \bigoplus_{\alpha \in \mathfrak{h}^*} \mathfrak{g}_\alpha$$

In this case the  $\mathfrak{g}$ -module  $M$  is assumed to be  $\mathfrak{h}$ -diagonalizable

$$(62) \quad M = \bigoplus_{\lambda \in \mathfrak{h}^*} M_\lambda, \quad \dim M_\lambda < \infty$$

where each weight space is finite dimensional. Here  $\mathfrak{h}$  is a Cartan subalgebra in the triangle decomposition  $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{h} \oplus \mathfrak{g}_+$ . It is known that  $M$  must be a direct sum of highest weight module  $Ind_{\mathfrak{g}_-}^{\mathfrak{g}} \mathbb{C}_\Lambda$  or a lowest weight module  $Ind_{\mathfrak{g}_+}^{\mathfrak{g}} \mathbb{C}_\Lambda$  (Verma modules) or an intermediate series defined by  $V_{a,b}$ ,  $a, b \in \mathbb{C}$ ,

$$(63) \quad V_{a,b} = \bigoplus_n \mathbb{C}v_n, \quad L_s.v_n = (as + b - n)v_{n+s}, \quad C.v_n = 0$$

where  $L_s$  are generators of the Virasoro algebra, see [F1] and [IK] for details.

**(5) Jantzen filtration and Shapovalov form:** Let  $(\mathfrak{g}, \mathfrak{h})$  be a lie algebra pair,  $\mathfrak{h}$  a Cartan subalgebra, with the anti-involution  $\sigma : \mathfrak{g} \rightarrow \mathfrak{g}$ . The context of this item is applicable to any  $Q$ -graded Lie algebra,

$$(64) \quad \mathfrak{g} = \bigoplus_{\beta \in Q} \mathfrak{g}_\beta, \quad \dim \mathfrak{g}_\beta < \infty$$

where  $Q$  is an abelian group. In our case  $Q = \mathfrak{h}^*$  is the root lattice. In particular the definitions are applicable to usual finite dimensional Lie algebras as well as their affine or Virasoro algebras, which are infinite dimensional. Write

$$(65) \quad \mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{h} \oplus \mathfrak{g}_+$$

The universal enveloping algebra of  $\mathfrak{g}$  is by definition the quotient of the tensor algebra  $T(\mathfrak{g}) = \bigoplus_n \mathfrak{g}^{\otimes n}$  by the ideal generated by  $x \otimes y - y \otimes x - [x, y]$ . By the Poincaré-Birkhoff-Witt theorem we have the decomposition

$$(66) \quad U(\mathfrak{g}) = U(\mathfrak{h}) \oplus \{\mathfrak{g}_- U(\mathfrak{g}) + U(\mathfrak{g}) \mathfrak{g}_+\}$$

Consider the projection

$$(67) \quad \pi : U(\mathfrak{g}) \rightarrow U(\mathfrak{h}) \cong S(\mathfrak{h})$$

with respect to this decomposition ( $S(\mathfrak{h})$  is the symmetric algebra of  $\mathfrak{h}$ ). The bilinear form

$$(68) \quad F : U(\mathfrak{g}) \times U(\mathfrak{g}) \rightarrow S(\mathfrak{h}), \quad F(x, y) = \pi(\sigma(x)y)$$

is called the Shapovalov form of  $\mathfrak{g}$ . It is a symmetric and contravariant form, that is

$$(69) \quad F(zx, y) = F(x\sigma(y))$$

The decomposition (55) implies similar decomposition on the universal enveloping algebra

$$(70) \quad U(\mathfrak{g}) = \bigoplus_{\beta} U(\mathfrak{g})_{\beta}, \quad U(\mathfrak{g})_{\beta} = \{x \in U(\mathfrak{g}) \mid [h, x] = \beta(h)x, \forall h \in \mathfrak{h}\}$$

Then one can show that for  $\beta_1 \neq \beta_2 \in Q$

$$(71) \quad F(x, y) = 0, \quad x \in U(\mathfrak{g})_{\beta_1}, \quad y \in U(\mathfrak{g})_{\beta_2}$$

For each  $\beta \in Q$  one can choose a basis of  $U(\mathfrak{g})_{-\beta}$  namely  $X_j$ ,  $j \in I$ . The determinant

$$(72) \quad D_{\beta} = \det(F(X_i, X_j))_{i,j \in I} \in S(\mathfrak{h})$$

is called a Shapovalov determinant of  $\mathfrak{g}$ .

The notion of contravariant bilinear form can be defined for any  $\mathfrak{g}$ -module  $M$ . A basic fact in this context is; any highest weight module  $M$  has a unique contravariant bilinear form  $\langle \cdot, \cdot \rangle : M \otimes M \rightarrow \mathbb{C}$  up to a constant ( $\langle g.x, y \rangle = \langle x, \sigma(g).y \rangle$ ,  $g \in U(\mathfrak{g})$ ). This fact can be easily checked for instance on Verma modules  $M(\lambda)$  as the unique irreducible quotients of these modules. In case of Verma modules  $M_{\lambda}$ , the radical of such form is the maximal proper submodule  $J(\lambda) \subset M(\lambda)$ .

The Jantzen filtration of a  $\mathfrak{g}$ -module is defined based on the Shapovalov form on  $U(\mathfrak{g})$ . Let  $R = \mathbb{C}[t]$  and  $\phi : R \rightarrow \mathbb{C}$  be the canonical map. Let  $\tilde{M}$  be a free  $R$ -module of rank  $r$  with a nondegenerate symmetric bilinear form

$$(73) \quad (\cdot, \cdot)_{\tilde{M}} : \tilde{M} \times \tilde{M} \rightarrow R$$

Set  $M = \phi\tilde{M} = M \otimes_R R/tR$ . Then  $M$  admits a symmetric bilinear form

$$(74) \quad (\phi v_1, \phi v_2) = \phi(v_1, v_2)_{\tilde{M}}$$

For  $m \in \mathbb{Z}_{\geq 0}$  set

$$(75) \quad \tilde{M}(m) = \{v \in \tilde{M} | (v, \tilde{M})_{\tilde{M}} \subset t^m R\} \xrightarrow{i_m} \tilde{M}$$

Then set  $M(m) = \text{Im}\phi(i_m)$ . It follows that

$$(76) \quad M = M(0) \supset M(1) \supset \dots$$

defines a filtration of  $\mathbb{C}$ -vector spaces called Jantzen filtration. This filtration enjoys of the following properties (cf. [IK]),

- $\bigcap_m M(m) = 0$
- $M(1) = \text{rad}(\cdot, \cdot)$
- there exists a symmetric bilinear form  $(\cdot, \cdot)_m$  on  $M(m)$  such that  $\text{rad}(\cdot, \cdot)_m = M(m+1)$ .

The procedure of defining the Jantzen filtration appears both in the context of vertex algebras and variations of Hodge structure. As explained before in the correspondence between these two sort of objects we expect that the filtrations also correspond to one another. The Jantzen filtration will correspond to the weight filtration in local systems of mixed Hodge structure. In the context of Hodge theory the Hodge filtration is defined via  $V$ -filtration of the associated Gauss-Manin system in a slightly similar methods.

**(6) Unitary (conformal) vertex operator algebras:** Let  $(V, Y, 1, w)$  be a vertex algebra and  $\phi : V \rightarrow V$  be an antilinear involution.  $(V, \phi)$  is called unitary if there exists a positive definite Hermitian form

$$(77) \quad (\cdot, \cdot) : V \times V \rightarrow \mathbb{C}$$

such that for  $a, u, v \in V$

$$(78) \quad (Y(e^{zL(1)}(-z^{-2})^{L(0)}a, z^{-1})u, v) = (u, Y(\phi(a), z)v),$$

where

$$(79) \quad Y(w, z) = \sum_n L(n)z^{-n-2}$$

In a unitary vertex operator algebra the positive definite Hermitian form is uniquely determined by  $(1, 1)$  via the properties of vertex algebra mentioned before.

In the Virasoro case  $V(c, h)$ , there exists a unique Hermitian form with

$$(80) \quad (1_{c,h}, 1_{c,h}) = 1, \quad (L_n u, v) = (u, L_{-n} v)$$

It is known that  $V(c, h)$  is unitary iff  $c \geq 1$ ,  $h \geq 0$  or  $c = c_m$ ,  $h = h_{r,s}^m$  where

$$(81) \quad c_m = 1 - \frac{6}{m(m+1)}, \quad h_{r,s}^m = \frac{r(m+1) - sm)^2 - 1}{4m(m+1)}$$

In the affine case  $V_{\mathfrak{g}}(k, \lambda)$  has a unique positive definite Hermitian form such that

$$(82) \quad (1, 1) = 1, \quad (xu, v) = -(u, \widehat{\omega}_0(x)v), \quad x \in \widehat{\mathfrak{g}}, \quad u, v \in L_{\mathfrak{g}}(k, \lambda)$$

where

$$(83) \quad \widehat{\omega}_0 : \widehat{\mathfrak{g}} \rightarrow \widehat{\mathfrak{g}}$$

is the Cartan involution. Then  $V_{\mathfrak{g}}(k, \lambda)$  with  $k \neq -h^\vee$  is a unitary vertex algebra iff  $k \in \mathbb{Z}^+$  and  $\lambda$  is a dominant integral weight satisfying  $(\lambda, \theta) \leq k$ .

In the Fock module case we reduce to  $M(1, \lambda) = U(\widehat{\mathfrak{h}})/J_\lambda$ . It is known that it is unitary iff  $(\alpha, \lambda) \geq 0$ , i.e  $\lambda$  be a dominant weight, [DX].

**(7) Conformal vertex algebras:** A vertex algebra  $V = \oplus_n V_n$  of central charge  $c$  is called conformal if it contains a vector  $w \in V_2$  (called conformal vector) such that the corresponding vertex operator  $Y(w, z) = \sum_n L_n z^{-n-2}$  satisfies

$$(84) \quad L_{-1} = T, \quad L_0|_{V_n} = n.Id, \quad L_2 w = \frac{1}{2}c|0\rangle$$

It follows that there is a homomorphism

$$(85) \quad Vir_c \rightarrow V, \quad L_{-2}1_c \mapsto w$$

In the Kac-Moody case the conformal vector (called Sugawara conformal vector) is given by

$$(86) \quad \frac{1}{2(k + h^\vee)} \sum_a (J_{-1}^a)^2 1_k$$

where  $J^a$  is an orthonormal basis of  $\mathfrak{g}$ . Thus a Kac-Moody algebra is conformal iff  $k \neq -h^\vee$ . In this case  $\widehat{\mathfrak{g}}$  is a module over Virasoro algebra, [F1], [F2].

### 3. VARIATION OF HODGE STRUCTURE

A variation of Hodge structure over a complex manifold  $S$  gives rise to a period map

$$\Phi : S \rightarrow \Gamma_{\mathbb{Z}} \backslash D$$

where  $S$  is a smooth base manifold and  $\Gamma$  is a discrete group.  $D$  is the period domain and it is known that it is a hermitian symmetric complex manifold. There are naturally defined Hodge bundles  $F^p$  of the Hodge structure on  $V$ , and also the endomorphism bundle associated to  $\mathfrak{g} = \text{End}(V)$  on  $D$ . The corresponding local systems are

$$(87) \quad \mathcal{V} := \Gamma \backslash (D \times \mathcal{V}), \quad \mathcal{G} := \Gamma \backslash (D \times \mathfrak{g})$$

respectively. One way to explain the complex structure on  $D$  is to embed it in its compact dual  $\check{D}$ , which is the set of all Hodge filtrations on  $V$  with the same Hodge numbers satisfying the first Riemann-Hodge bilinear relation.  $\check{D}$  is a homogeneous complex manifold. There are  $G_{\mathbb{C}}$ -homogeneous vector bundles

$$(88) \quad F^p \rightarrow \check{D}$$

called Hodge bundles whose fiber at a given point  $F^\bullet$  is  $F^p$ . Over  $D \subset \check{D}$  we have  $V^{p,q} = F^p / F^{p+1}$ , which are homogeneous vector bundles for the action of  $G_{\mathbb{R}}$ . They are Hermitian vector bundles with  $G_{\mathbb{R}}$ -invariant Hermitian metric given in each fiber by the polarization form. The space of functions on  $D$  can be identified with the  $\Gamma_{\mathbb{Z}}$ -automorphic functions on  $D$ .

**(1) Variation of mixed Hodge structure:** A polarized variation of mixed Hodge structure over the punctured disc  $\Delta^*$  consists of the 5-tuple  $(\mathcal{V}, F^\bullet, W_\bullet, \nabla, P)$  where

- $\mathcal{V}$  is a local system of  $\mathbb{Q}$ -vector spaces on  $\Delta^*$ .
- $W_\bullet$  is an increasing filtration on  $\mathcal{V}$  by sub-local systems of  $\mathbb{Q}$ -vector spaces.
- $F^\bullet = (F^i)$  is a decreasing filtration on the vector bundle  $\mathcal{V} \otimes_{\mathbb{Q}} \mathcal{O}_{\Delta^*}$  by holomorphic sub-bundles.
- $\nabla : \mathcal{V} \otimes_{\mathbb{Q}} \mathcal{O}_{\Delta^*} \rightarrow \mathcal{V} \otimes_{\mathbb{Q}} \Omega_{\Delta^*}^1$  is a flat connection satisfying Griffiths transversality;



$$(89) \quad \nabla(F^i) \subset F^{i-1} \otimes \Omega_{\Delta^*}^1$$

–  $P : \mathcal{V} \times \mathcal{V} \rightarrow \mathbb{Q}$  is a flat pairing inducing a set of rational non-degenerate bilinear forms  $P_k : Gr_W^k \mathcal{V} \otimes Gr_W^k \mathcal{V} \rightarrow \mathbb{Q}$  such that the triple

$$(90) \quad (Gr_W^k \mathcal{V}, F^\bullet Gr_W^k, P_k)$$

defines pure polarized variation of Hodge structure on  $\Delta^*$ .  
which we briefly mention as  $\mathcal{H}$ .

Let  $V$  a Hodge structure with an exhaustive decreasing Hodge filtration  $F^p$ . Regard a locally free sheaf  $\xi(V, F)$  over  $\mathbb{C}$  as the submodule of  $V \otimes \mathbb{C}[t, t^{-1}]$  generated by  $t^{-p}F^p$ . Given a real structure one can glue  $\xi(V, F)$  and  $\xi(V, \overline{F})$  using the involution  $t \rightarrow (\bar{t})^{-1}$  to obtain a locally free sheaf  $\xi(V, F, \overline{F})$  on  $\mathbb{P}^1$  with the action of  $\mathbb{C}^*$  and antilinear involution. This procedure may be explained as follows. A variation of polarized Hodge structure of weight  $k$  provides a 4-tuple  $(H, F, \nabla, P)$  where

$$(91) \quad \nabla : H \rightarrow H \otimes z^{-1} \Omega_{\Delta^*}(\log 0)$$

is a flat connection and a  $(-1)^k$ -symmetric non-degenerate and flat pairing

$$(92) \quad P : H \times j^* H \rightarrow \mathcal{O}_{\Delta^*}, \quad j : z \mapsto -z$$

The bilinear form  $P$  induces a non-degenerate symmetric pairing

$$(93) \quad z^{-k} P : H/zH \times H/zH \rightarrow \mathbb{C}$$

The Hodge filtration can be explained as follows. Lets  $V$  be the Kashiwara-Malgrange filtration on the mixed Hodge module associated to  $H$ . and suppose  $(H, \nabla)$  is regular singular. Then  $H \hookrightarrow V^{>-\infty}$ . For  $\alpha \in (0, 1]$ , define (cf. [DW]),

$$(94) \quad F^p H_\lambda := z^{\beta + \frac{N}{2\pi i}} Gr_V^{-\beta} H$$

We will identify the variation of polarized Hodge structures with their associated polarizable Hodge module via the Riemann-Hilbert correspondence. This correspondence has also been studied in a more systematic way by M. Saito [SA] via the relation (97) in the next section. A parallel insight toward this correspondence can be given by the non-abelian Hodge theorem via the Higgs field associated to the flat connections. In this case the Hodge filtration

may be compared with the Harder-Narasimhan filtration. We discuss this in the following sessions.

- (2) **Polarizable Mixed Hodge modules:** Let  $X$  be a complex algebraic variety and denote by  $MHM(X)$ , the abelian category of Mixed Hodge Modules on  $X$ .  $MHM(X)$  is equipped with a forgetful functor

$$(95) \quad \text{rat} : MHM(X) \rightarrow \text{Perv}(\mathbb{Q}_X)$$

which assigns the underlying perverse sheaf/ $\mathbb{Q}$ . Sometimes the above objects is understood as elements in  $D^bMHM(X)$  and  $D_c^b(\mathbb{Q}_X)$  respectively, and the same for the functor  $\text{rat}$ . When  $X$  is smooth, then a mixed Hodge module on  $X$  determines a 4-tuple  $(M, F, K, W)$  where  $M$  is a holonomic  $D$ -module with a *good* filtration  $F$  and, with rational structure

$$(96) \quad \text{DR}(M) \cong \mathbb{C} \otimes K \in \text{Perv}(\mathbb{C}_X)$$

for a perverse sheaf  $K$ , and  $W$  is a pair of weight filtrations on  $M$  and  $K$  compatible with  $\text{rat}$  functor.  $\text{DR}$  denotes the de Rham functor shifted by the  $\dim(X)$ . The de Rham functor is dual to the solution functor. If  $X = pt$ , Then,  $MHM(pt)$  is exactly all the polarizable mixed Hodge structures.

A MHM always has a weight filtration  $W$ , and we say it is *pure of weight  $n$* , if  $Gr_k^W = 0$  for  $k \neq n$ . Normally, the filtration  $W$  is involved with a nilpotent operator on  $M$  or the underlying variation of a mixed Hodge structure. A mixed Hodge modules (def.) is obtained by successive extensions of pure one. If the support of a pure Hodge module as a sheaf is irreducible such that no sub or quotient module has smaller support, then we say the module has *strict support*. Any pure Hodge module will have a unique decomposition into pure modules with different strict supports, known as Decomposition Theorem. A pure Hodge module is also called polarizable HM.  $MH_Z(X, n)^p$  will denote the category of pure Hodge modules with strict support  $Z$ . An  $M \in HM_Z(X, n)$  determines a polarizable variation of Hodge structure. The converse of this fact is also true, that variation of Hodge structures determine a MHM. Thus;

$$(97) \quad MH_Z(X, n)^p \simeq VHS_{gen}(Z, n - \dim Z)^p$$

The right side means polarizable variations of Hodge structure of weight  $n - \dim Z$  defined on a non-empty smooth sub-variety of  $Z$ . Equation (97), explains a deep non-trivial fact about regular holonomic  $D$ -modules, their

underlying perverse sheaves and their polarizations. It may also be interpreted as an analogue of Riemann-Hilbert correspondence between mixed Hodge modules and their underlying perverse sheaves, [R].

- (3) **Higgs bundles and non-abelian Hodge theorem:** Suppose  $X$  is smooth and projective over  $\mathbb{C}$ . A harmonic bundle on  $X$  is a  $C^\infty$ -vector bundle  $E$  with differential operators  $\partial$  and  $\bar{\partial}$  and algebraic operators  $\theta$  and  $\bar{\theta}$  such that the following holds: There exists a metric  $h$  so that  $\partial + \bar{\partial}$  is a unitary connection and  $\theta + \bar{\theta}$  is self adjoint. And if

$$(98) \quad \nabla = \partial + \bar{\partial} + \theta + \bar{\theta}, \quad \nabla'' = \bar{\partial} + \theta$$

then  $\nabla^2 = \nabla''^2 = 0$ . With these conditions  $(E, D)$  is a vector bundle with flat connection, and  $(E, \bar{\partial}, \theta)$  is a Higgs bundle, i.e a holomorphic vector bundle with holomorphic section

$$(99) \quad \theta \in H^0(\text{End}(E) \otimes \Omega_X^1), \quad \theta \wedge \theta = 0$$

A Higgs bundle is stable (resp. semistable) if for any coherent subsheaf  $F \subset E$  preserved by  $\theta$  we have

$$(100) \quad \deg(F)/\text{rank}(F) < \deg(E)/\text{rank}(E) \quad (\text{resp. } \leq)$$

There is a natural equivalence between the categories of harmonic bundles on  $X$  and semisimple flat bundles (or representations of  $\pi_1(X)$ ). There is also a natural equivalence between the categories of harmonic bundles and direct sum of stable Higgs bundles with vanishing Chern class. The resulting correspondence between representations and Higgs bundles can be extended to an equivalence between the category of all representations of  $\pi_1(X)$  and all semistable Higgs bundles with vanishing Chern classes. This statement is referred to as the non-abelian Hodge theorem.

There is a natural  $\mathbb{C}^*$  action on the category of semistable Higgs bundles with vanishing Chern classes, denoted

$$(101) \quad t : (E, \theta) \mapsto (E, t\theta)$$

The semistable representations which are fixed by this action are exactly complex variations of Hodge structure. A representation  $\rho$  of  $\pi_1(X)$  is called rigid if any nearby representation is conjugate to it. The correspondence described above is continuous on the moduli of semisimple representations. It follows that if a semisimple representation is called rigid it must be fixed by  $\mathbb{C}^*$  and it comes from a complex variation of Hodge structure. In this

case there is a  $\mathbb{Q}$ -variation of HS  $V_{\mathbb{Q}}$  such that  $\varrho$  is a direct factor of the monodromy representation of  $V_{\mathbb{Q}} \otimes \overline{\mathbb{Q}}$  (the monodromy is a sum of conjugates of  $\varrho$ ), [S].

Let  $M_{Dol}(G)$ ,  $M_{DR}(G)$ ,  $M_B(G)$  denote the moduli space of Higgs bundles of degree zero, local systems, and representations of  $\pi_1(X)$  respectively. We will denote the smooth loci of these varieties by the superscript *reg*, as  $M_{Dol}^{reg}, \dots$ , and we usually omit the script *reg* future on. The non-abelian Hodge theorem gives a diffeomorphism

$$(102) \quad \tau : M_{Dol}(G) \cong M_{DR}(G)$$

The Riemann-Hilbert correspondence between bundles with integrable connection and representations provides isomorphisms

$$(103) \quad M_{DR}(G) \cong M_B(G)$$

see also [S], [D]. A systematic study on the inter-relation of between the Higgs fields of Higgs bundles and the system of Hodge bundles in VHS can be found in [P]. The corollary is a unipotent variation of mixed Hodge structure defines a Higgs field  $\theta$  which is flat relative to  $\nabla$  and  $\partial + \bar{\partial}$ . In this case the invariance under the  $\mathbb{C}^*$ -action in (101) explains the complex variation of Hodge filtration, see [P] for details.

#### 4. RELATING HODGE STRUCTURES TO VERTEX ALGEBRAS

Let  $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g}_{\mathbb{R}} \otimes \mathbb{C} = Lie(G_{\mathbb{R}} \otimes \mathbb{C})$  be a complex semi-simple Lie algebra,  $\mathfrak{h} = \mathfrak{t} \otimes \mathbb{C}$  a Cartan subalgebra,  $\mathfrak{t} = Lie(T)$ , and  $K_{\mathbb{C}}$  a complex Lie group corresponding to the unique maximal compact subgroup  $K \subset G_{\mathbb{R}}$ . We denote  $U(\mathfrak{g}_{\mathbb{C}})$  to be the universal enveloping algebra of  $\mathfrak{g}_{\mathbb{C}}$ . We assume the action of  $K_{\mathbb{C}}$  will be locally finite, and its differential agrees with the corresponding subspace of  $U(\mathfrak{g}_{\mathbb{C}})$ . One may match these data with the case  $D = G_{\mathbb{R}}/H$  is a general period domain sitting in the diagram

$$(104) \quad \begin{array}{ccc} G_{\mathbb{R}}/T & \longrightarrow & G_{\mathbb{C}}/B \\ \downarrow & & \downarrow \\ D = G_{\mathbb{R}}/H & \longrightarrow & G_{\mathbb{C}}/P = \check{D} \end{array}$$

with  $T$  a maximal torus,  $B$  a Borel subgroup, and horizontal arrows to be inclusions. Let  $U(\mathfrak{h}_{\mathbb{C}})$  be the universal enveloping algebra of  $\mathfrak{h}$ . The Weyl group  $W$  of  $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$  acts on  $U(\mathfrak{h}_{\mathbb{C}})$  and gives an isomorphism

$$(105) \quad HC : Z(\mathfrak{g}_{\mathbb{C}}) \xrightarrow{\cong} U(\mathfrak{h}_{\mathbb{C}})^W$$

where the upper-index means the elements fixed by  $W$ . Using the isomorphism (105) one can assign to each positive root  $\mu \in \mathfrak{h}_{\mathbb{C}}^*$  the homomorphism

$$(106) \quad \chi_{\mu} : Z(\mathfrak{g}_{\mathbb{C}}) \rightarrow \mathbb{C}, \quad z \mapsto HC(z)(\mu)$$

is called the infinitesimal character associated to  $\mu$ . A result of Harish-Chandra says that any character of  $Z(\mathfrak{g}_{\mathbb{C}})$  is an infinitesimal character, and

$$(107) \quad \chi_{\mu} = \chi_{\mu'} \quad \Leftrightarrow \quad \mu = w(\mu'), \quad w \in W$$

We will use the above set up in our construction of correspondence between Hodge bundles and vertex algebras. In fact our correspondence uses a generalization of (105). We will do this step by step as follows.

**(1) Beilinson-Bernstein correspondence:** Let  $G$  be a connected, complex reductive algebraic group  $G$  defined over  $\mathbb{R}$ . Let  $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n} \subset \mathfrak{g} = Lie(G)$  be the Borel subalgebra, with unipotent radical

$$(108) \quad \mathfrak{n} = \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}^{-\alpha}$$

and Cartan algebra  $\mathfrak{h}$ . Any  $\lambda$  in the weight lattice  $\Lambda \subset \mathfrak{h}_{\mathbb{R}}^*$  lifts to an algebraic character  $e^{\lambda}$  of the Borel subgroup  $B \subset G$  corresponding to  $\mathfrak{b}$ . To this character there corresponds a unique  $G$ -equivariant line bundle  $\mathcal{L} \rightarrow X = G/B$ . The group  $B$  acts as  $e^{\lambda}$  on the fibers. Therefore  $\Lambda$  can be regarded as the group of  $G$ -equivariant line bundles via  $\lambda \mapsto \mathcal{L}_{\lambda}$ . Let  $\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha \in \frac{1}{2}\Lambda$ . Then

$$(109) \quad \mathcal{L}_{-2\rho} \cong \bigwedge^n T^*X$$

If  $G$  is simply connected then  $\rho \in \Lambda$  and  $\mathcal{L}_{-2\rho}$  has a well defined square root. Define

$$(110) \quad D_{\lambda} = \mathcal{O}(\mathcal{L}_{\lambda-\rho}) \otimes D_X \otimes \mathcal{O}(\mathcal{L}_{\rho-\lambda})$$

The Lie algebra  $\mathfrak{g}$  acts by infinitesimal translation on sections of  $\mathcal{L}_{\lambda-\rho}$ . Thus  $\mathfrak{g} \hookrightarrow \Gamma D_{\lambda}$  which induces

$$(111) \quad U(\mathfrak{g}) \rightarrow \Gamma D_{\lambda}$$

where the center  $Z(\mathfrak{g})$  of  $U(\mathfrak{g})$  acts via the infinitesimal character  $\chi_\lambda$ . Therefore we get a homomorphism

$$(112) \quad U_\lambda(\mathfrak{g}) = U(\mathfrak{g}) / \ker(\chi_\lambda) \rightarrow \Gamma D_\lambda$$

which is compatible with degree filtration. The Beilinson-Bernstein theorem asserts that the map (112) is an isomorphism. Let  $Mod(U_\lambda(\mathfrak{g}))_{fg}$  be the category of finitely generated  $U_\lambda(\mathfrak{g})$ -modules or equivalently the category of finitely generated  $U_\lambda(\mathfrak{g})$ -modules on which  $Z(\mathfrak{g})$  acts via the character  $\chi_\lambda$ . Also let  $Mod(D_\lambda)_{coh}$  refer to the category of coherent  $D_\lambda$ -modules. Define the two functors

$$(113) \quad \Delta : Mod(U_\lambda(\mathfrak{g}))_{fg} \rightarrow Mod(D_\lambda)_{coh}, \quad \Delta(M) = M \otimes_{U_\lambda(\mathfrak{g})} D_\lambda$$

$$(114) \quad \Gamma : Mod(D_\lambda)_{coh} \rightarrow Mod(U_\lambda(\mathfrak{g}))_{fg}, \quad \Gamma(M) = H^0(X, M)$$

A theorem by Beilinson and Bernstein asserts that when  $\lambda$  is a regular and integrally dominant weight the above functors define an equivalence of categories

$$(115) \quad Mod(U_\lambda(\mathfrak{g}))_{fg} \cong Mod(D_\lambda)_{coh}$$

In the same case for  $\lambda$ , a similar isomorphism will hold between the category of Harish-Chandra modules

$$(116) \quad HC(\mathfrak{g}, K)_\lambda \cong HC(D_\lambda, K)$$

where in the right hand side we mean the category of  $D_\lambda$ -modules with a compatible  $K$ -action.

If we consider the polarization of  $D$ -modules as a Hermitian duality,

$$(117) \quad P : M \times \overline{M} \rightarrow C^\infty(X_\mathbb{R}), \quad \text{bilinear over } D_\lambda \times \overline{D}_{-\lambda}$$

In our settings,  $M \mapsto \overline{M}$  defines a bijection  $Mod(D_{X,\lambda})_{rh} \cong Mod(D_{\overline{X},-\lambda})_{rh}$ . We sometimes forget the subscripts  $X$  or  $\overline{X}$  and simply write  $D_\lambda$  and  $D_{-\lambda}$ . The pairing  $P$  should be understood as a pairing in the form  $(\sigma, \tau) = \int_X \langle \sigma, \bar{\tau} \rangle$  where  $\langle \cdot, \cdot \rangle$  is flat hermitian pairing on the underlying vector bundles, and it is  $\mathfrak{u}_\mathbb{R}$ -invariant, where  $\mathfrak{u}$  is the compact form of  $\mathfrak{g}$  defined via a Cartan involution. In the correspondence (113-114) the flat bilinear form  $P$  in (117) corresponds

to the Shapovalov form and the weight filtration to the Jantzen filtration, [SV].

The Beilinson-Bernstein correspondence (113-114) is a first step in our intent to correspond the variation of mixed Hodge structure to vertex algebras. In fact it is the desired correspondence over flag varieties. We propose to do this on an arbitrary base or at least a one dimensional base manifold. Of course analysing  $D$ -modules over flag varieties seems much easier than other varieties. Thus we will need to make specific considerations to extend the localization theorem of Beilinson-Bernstein. This we will do in the preceding sections.

- (2) **Localization Functor:** We need to formulate the Beilinson-Bernstein correspondence in a slightly more general language. Let  $G$  be a connected simple Lie group over  $\mathbb{C}$ . Assume  $LG = G((t))$  is the Lie group of  $L\mathfrak{g} = \mathfrak{g}((t))$ . Let  $X$  be a smooth projective algebraic curve over  $\mathbb{C}$  and  $p \in X$  a point. Let  $P$  be a principal  $G$ -bundle over  $X$ . Let

$$(118) \quad \mathfrak{g}_P = P \times_G \mathfrak{g}$$

be the vector bundle associated to the adjoint representation of  $G$ . Let  $\mathfrak{g}_{out}^P$  be the Lie algebra of sections of  $\mathfrak{g}_P$  around  $p \in X$ , and let  $G_{out}$  be the Lie group of  $\mathfrak{g}_{out}$ . There is a natural embedding

$$(119) \quad \mathfrak{g}_{out}^P \hookrightarrow L\mathfrak{g}$$

which can be lifted to  $\mathfrak{g}_{out}^P \rightarrow \widehat{\mathfrak{g}}$ . Denote by  $\mathcal{O}^0$  the category of  $\widehat{\mathfrak{g}}$ -modules where the Lie subalgebra  $\mathfrak{g}_{in} = \mathfrak{g}[[t]]$  acts locally finite. The modular functor assigns to a module

$$(120) \quad M \longmapsto M/\mathfrak{g}_{out}^P M$$

The dual space of  $M/\mathfrak{g}_{out}^P M$  is called the space of conformal blocks. The localization functor of Beilinson and Bernstein assigns to  $M$  a  $D$ -module on the homogeneous space

$$(121) \quad \mathcal{M} = LG/G_{out}$$

For any integer  $k$  define a line bundle  $\mathcal{L}^k$  on  $\mathcal{M}$  together with a homomorphism from  $\widehat{\mathfrak{g}}$  to the Lie algebra of infinitesimal symmetries of  $\mathcal{L}^k$ . This gives a homomorphism from  $U_k(\widehat{\mathfrak{g}})$  to the algebra  $D_k$  of global differential operators on  $\mathcal{L}^k$ . Thus for any  $\widehat{\mathfrak{g}}$ -module  $M$  of level  $k$  we can define a left  $D_k$ -module

$$(122) \quad \Delta(M) = D_k \otimes_{U(\widehat{\mathfrak{g}})} M$$

The fiber of  $\Delta(M)$  at  $P$  is  $M/\mathfrak{g}_{out}^P M$ , [F2]. The map  $\Delta$  appearing here is an analogue of (113) and generalize that over affine algebras. The adjoint functor is the global section functor. The same as previous section the Shapovalov form of  $M$  correspond to the flat Hermitian pairing in the right side. We will employ the map

$$(123) \quad \Delta : \widehat{\mathfrak{g}}\text{-Mod} \longrightarrow D_{\mathcal{M}}\text{-Mod}$$

which is a generalization of its analogue (113), in the procedure of geometric Langlands correspondence in Session (6).

**(3)  $\mathfrak{g}$ -opers on the punctured disc:** We first define  $GL(n)$ -opers, that is a pair  $(F_{\bullet} \subset E, D)$ , where  $E$  is a vector bundle of rank  $n$  equipped with

- a complete flag  $(0) \subset F_1 \subset \dots \subset F_n = E$ ,  $rank(F_i) = i$ ,
- a holomorphic connection  $D : E \rightarrow E \otimes K$  necessarily flat such that,
- Griffiths transversality;  $D : F_i \rightarrow F_{i+1} \otimes K$ ,
- non-degeneracy (strictness)  $Gr_i D : Gr_i^F E \cong Gr_{i+1}^F E \otimes K$ .

An  $SL(n)$ -oper is a  $GL(n)$ -oper such that  $\det(E) = \mathcal{O}$  and  $D$  induces trivial connection  $d$  on  $\det E$ .

Now assume  $G$  is a simple complex group, with  $B \subset G$  a fixed Borel subgroup,  $N = [B, B]$  its unipotent radical,  $H = B/N$  the Cartan subgroup, and  $Z = Z_G$  the center. The corresponding Lie algebras are  $\mathfrak{n} \subset \mathfrak{b} \subset \mathfrak{g}$ ,  $\mathfrak{t} = \mathfrak{b}/\mathfrak{n}$ . If  $P_B$  is a holomorphic principal  $B$ -bundle,  $P_G$  the induced  $G$ -bundle and  $Conn(P_G)$  the sheaf of connections. Define the projection

$$(124) \quad c : Conn(P_G) \rightarrow \mathfrak{g}/\mathfrak{b}_P \otimes K$$

by requiring  $c^{-1}(0) = Conn(P_B)$  and  $c(D + \nu) = c(D) + [\nu]$  where  $\nu$  is a section of  $\mathfrak{g}_P \otimes K$  and  $[\nu]$  is its image in  $\mathfrak{g}/\mathfrak{b}_P \otimes K$ . We will consider the class

$$(125) \quad c(D) \in H^0(\mathfrak{g}/\mathfrak{b}_P \otimes K)$$

by taking locally any flat  $B$ -connection  $D_B$ , and then glue the local sections  $[D - D_B]$ . Then, a  $G$ -oper on  $X$  is a pair  $(P_B, D)$ ,  $D \in H^0(Conn(P_G))$  such that

$$- c(D) \in H^0((\mathfrak{g}_{-1})_P \otimes K) \subset H^0(\mathfrak{g}/\mathfrak{b}_P \otimes K)$$



- For any simple negative root  $\alpha$  the component  $c(D)_\alpha \in H^0(\mathfrak{g}/\mathfrak{b}_P \otimes K)$  is nowhere vanishing.

The meaning of the conditions is that the connection  $D$  preserves the flag corresponded to the Borel subgroup  $B$  via the Griffiths transversality. By definition a  $\mathfrak{g}$ -oper is a  $G$ -oper where  $G$  is the group of inner automorphisms of  $\mathfrak{g}$ .

For  $GL(n)$  the oper condition implies that if  $E_U \cong \mathcal{O}^n$  is a trivialization compatible with the flag on an open chart  $U$ , then one can write the flat connection

$$(126) \quad d + \begin{pmatrix} * & * & \dots & * & * \\ \times & * & \dots & * & * \\ 0 & \times & \dots & * & * \\ \dots & \dots & \dots & \dots & * \\ 0 & 0 & \dots & \times & * \end{pmatrix} dt$$

Then a  $B$ -Gauge equivalence class of a  $GL_n$ -oper has a unique representative of the form

$$(127) \quad d + \begin{pmatrix} a_1 & a_2 & \dots & \dots & a_n \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & 0 \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix} dt$$

When we discuss about  $\mathfrak{g}$ -opers we automatically are considering their  $B$ -gauge equivalence classes, [D]. Lets restrict to opers on the punctured disc. By definition the space of  $\mathfrak{g}$ -opers on the punctured disc  $D^\times$  is

$$(128) \quad \mathcal{O}_{\mathfrak{p}_{\mathfrak{g}}}(D^\times) = \left\{ \sum_i \psi_i X_{-\alpha_i} + v \mid 0 \neq \psi_i \in \mathbb{C}((t)), v \in \mathfrak{b}((t)) \right\} / B((t))$$

where  $\alpha_i$  are the set of positive simple roots of  $\mathfrak{g}$  with respect to  $B$ . The action of  $B((t))$  is via the guege transformation by

$$(129) \quad g.D = Ad(g)D - (\partial_t g)g^{-1}$$

An oper on the punctured disc is called nilpotent if its connection has regular singularity at the origin with unipotent monodromy.  $\mathfrak{g}$ -opers as gauge equivalence classes of flat connections can be compared with mixed Hodge modules. The fact is  $\mathfrak{g}$ -opers on one hand are connected to  $\widehat{\mathfrak{g}^L}$ -modules and on the other hand to twisted  $D$ -modules with flat connections. In fact to

make a correspondence between certain mixed Hodge modules and opers one needs to specific considerations. We will explain this in the preceding sections. In this correspondence the filtration  $F_i$  corresponds to the Hodge filtration. it may also be translated to the language of Higgs fields. To any system of Hodge bundles one can associate a Higgs field by (98) which is stable under  $\mathbb{C}^*$ -action in (101). In this context the filtration  $F_i$  corresponds to the Harder-Narasimhan filtration. In this language the moduli  $\mathcal{M}$  defined in (121) is replaced by a moduli of Higgs bundles, cf. [D] prop. 4.2.1, see also [BD] and [F2].

- (4)  **$\widehat{\mathfrak{g}}$ -modules associated to opers:** Let  $\mathfrak{g}$  be a finite dimensional semisimple Lie algebra and  $\widehat{\mathfrak{g}}$  its affine Lie algebra. Define

$$(130) \quad \tilde{U}(\widehat{\mathfrak{g}}) := \varprojlim U(\widehat{\mathfrak{g}})/U(\widehat{\mathfrak{g}})(\mathfrak{g} \otimes t^n \mathbb{C}[t])$$

It follows from a formula of Kac-Kazhdan for the determinant of the Shapovalov form that the module  $V_k(\widehat{\mathfrak{g}})$  contains null vectors other than the highest weight vector  $v_k$  only if  $k = -h^\vee$  (the critical level). The space of null vectors  $\mathfrak{z}(\mathfrak{g})$  of  $V := V_{-h^\vee}$  is isomorphic to  $End_{\widehat{\mathfrak{g}}}(V)$ . To each vector  $v \in V$  we can associate a power series

$$(131) \quad v \longmapsto Y(v, z) = \sum_m v_m z^m$$

as in (27). The coefficient of these power series are elements of  $\tilde{U}_{-h^\vee}(\widehat{\mathfrak{g}}) = \tilde{U}(\widehat{\mathfrak{g}})/(K + h^\vee)$ . They span a Lie subalgebra  $\tilde{U}_{-h^\vee}(\widehat{\mathfrak{g}})_{loc}$ . For example

$$(132) \quad A \in \mathfrak{g} \longmapsto Y((A \otimes t^{-1})v, z) = A(z) = \sum_n (A \otimes t^n) z^{-n-1}$$

This shows  $\widehat{\mathfrak{g}} \subset \tilde{U}_{-h^\vee}(\widehat{\mathfrak{g}})_{loc}$ . Let  $Z(\widehat{\mathfrak{g}})$  be the center of  $\tilde{U}_{-h^\vee}(\widehat{\mathfrak{g}})_{loc}$ . One can show that

$$(133) \quad x \in \mathfrak{z}(\widehat{\mathfrak{g}}) \quad \Leftrightarrow \quad Y(x, z) \in Z(\widehat{\mathfrak{g}})$$

and all the elements of  $Z(\widehat{\mathfrak{g}})$  can be obtained in this form. A basic example of this is the Casimir element

$$(134) \quad S = \frac{1}{2} \sum_n (J_a \otimes t^{-1})^2 \in \mathfrak{z}(\widehat{\mathfrak{g}})$$

where  $J_a$ ,  $a = 1, \dots, \dim \mathfrak{g}$  is an orthonormal basis of  $\mathfrak{g}$  with respect to the invariant bilinear form. The coefficient  $S_n$  of the power series

$$(135) \quad Y(S, z) = \sum_n S_n z^{-n-2} = \frac{1}{2} \sum_a : J_a(z)^2 :$$

are called the Sugawara operators and lie in  $Z(\widehat{\mathfrak{g}})$ . Let

$$(136) \quad M_{\chi, k} = U_k(\widehat{\mathfrak{g}}) \otimes_{U(\widehat{\mathfrak{b}}_+)} \mathbb{C}_\chi, \quad \tilde{\mathfrak{b}}_+ = (\mathfrak{b}_+ \otimes 1) \oplus (\mathfrak{g} \otimes t\mathbb{C}[[t]])$$

be a Verma module over  $\widehat{\mathfrak{g}}$  and  $\chi \in \mathfrak{h}^*$  with highest weight vector  $v_{\chi, k}$ . Denote by  $\mathfrak{g}^L$  the Langlands dual of  $\mathfrak{g}$  obtained by exchanging the role of roots and co-roots. By classical theorems the center  $Z(\widehat{\mathfrak{g}})$  is isomorphic to  $W(\mathfrak{g}^L)$ ; the space of local functionals on  $\mathcal{Op}(\mathfrak{g}^L)$ . Let  $\rho \in \mathcal{Op}(\mathfrak{g}^L)$  be  $\mathfrak{g}^L$ -oper on the punctured disc. Then by what was said  $\rho$  defines a central character  $\tilde{\rho} : Z(\widehat{\mathfrak{g}}) \rightarrow \mathbb{C}$ . Then we associate the  $\widehat{\mathfrak{g}}$ -module  $M_\chi^\rho = M_{\chi, -h^\vee} / \ker \tilde{\rho}$  to the  $\mathfrak{g}^L$ -oper  $\rho$ , [F2]. Therefore we obtain a map

$$(137) \quad \mathcal{Op}(\mathfrak{g}^L) \longrightarrow \text{Mod}(\widehat{\mathfrak{g}}), \quad \rho \longmapsto M_\chi^\rho = M_{\chi, -h^\vee} / \ker \tilde{\rho}$$

This is part of the correspondence between the connections and  $\widehat{\mathfrak{g}}$ -modules. The assignment or interpretations of  $\mathfrak{g}^L$ -opers as characters on the center of the dual affine algebra  $Z(\widehat{\mathfrak{g}})$  is quite fundamental in this context.

- (5) **Wakimoto modules:** Our explanation of Wakimoto modules is quite brief, however we need this step in order to complete our correspondence. Assume we are given a linear function  $\chi : \mathfrak{h}((t)) \rightarrow \mathbb{C}$ . We extend it trivially to  $\mathfrak{n}((t))$  and obtain a linear function on  $\mathfrak{b}_-((t))$  also denote it by  $\chi$ . Now instead of considering  $Ind_{\mathfrak{b}_-((t))}^{\mathfrak{g}((t))} \mathbb{C}_\chi$  we will consider the semi-infinite induction [FF], [F4]. The resulting module is a module over the central extension of  $\mathfrak{g}((t))$  i.e over  $\widehat{\mathfrak{g}}$  of critical level where the vacuum is killed by  $t\mathfrak{g}[t] \oplus \mathfrak{n}_-$ . The parameters of the module will no longer behave as functionals on  $\mathfrak{h}((t))$ , but as connections on the  $H^L$ -bundle  $\Omega^i$  (sheaf of  $i$ -forms). They are precisely elements of the space  $Conn_{\mathfrak{g}^L}(D^\times)$  ( $D^\times$  is the punctured disc). We obtain a family of smooth representations of  $\tilde{U}_{\kappa_c} \widehat{\mathfrak{g}}$  parametrized by  $Conn_{\mathfrak{g}^L}(D^\times)$ . For  $\chi \in Conn_{\mathfrak{g}^L}(D^\times)$  denote the corresponding Wakimoto module by  $W_\chi$ . The center  $Z_{\kappa_c} \widehat{\mathfrak{g}}$  acts on  $W_\chi$  according to a character. The corresponding point in  $Spec(Z_{\kappa_c} \widehat{\mathfrak{g}}) = \mathcal{Op}_{\mathfrak{g}^L}(\Delta^*)$  is denoted by  $\mu(\chi)$ . We obtain a map

$$(138) \quad \mu : Conn_{\mathfrak{g}^L}(D^\times) \rightarrow \mathcal{Op}_{\mathfrak{g}^L}(D^\times)$$

called the Miura transformation. Regarding the context of opers in Section (3) the Miura transformation should be understood as

$$(139) \quad \partial_t - \begin{pmatrix} 0 & q_1(t) & \dots & & q_{n-1}(t) \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & 0 \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix} dt \longmapsto \partial_t - \begin{pmatrix} \chi_1(t) & 0 & \dots & \dots & 0 \\ 1 & \chi_2(t) & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & 0 \\ 0 & 0 & \dots & 1 & \chi_n(t) \end{pmatrix} dt$$

This amounts to the following splitting of the differential operator

$$(140) \quad \partial_t^n - q_1(t)\partial_t^{n-1} - \dots - q_{n-1}(t) = (\partial_t - \chi_1(t))\dots(\partial_t - \chi_n(t))$$

Miura transformation is an analogue of Harish-Chandra homomorphism

$$(141) \quad HC : Z(\mathfrak{g}) \rightarrow \mathbb{C}[\mathfrak{h}^*]^W$$

The fiber of the Miura transformation over a nilpotent oper is the variety of all Borel subalgebras  $\mathfrak{g}^L$  containing that oper. It is called Springer fiber over the nilpotent oper. For example the Springer fiber over 0 is the flag variety of  $\mathfrak{g}^L$ , [F3]. The composition of (137) and (138) gives a map

$$(142) \quad Conn_{\mathfrak{g}^L}(D^\times) \longrightarrow \mathcal{Op}(\mathfrak{g}^L) \longrightarrow Mod(\widehat{\mathfrak{g}})$$

which is the way we correspond  $D$ -modules of flat connections to  $\widehat{\mathfrak{g}}$ -modules or vertex algebras. A variation of MHS on the punctured disc can be interpreted as a flat connection that is regarded as an element of  $Conn_{\mathfrak{g}^L}(D^\times)$  via the action of  $\mathfrak{g}^L$  coming from the internal symmetries of the Hodge structure. In many cases the Lie algebra action is paired with a compatible action of the Lie group  $K$  to build up a Harish-Chandra pair  $(\mathfrak{g}, K)$ .

- (6) **Geometric Langlands correspondence:** A more solid way to reformulate the localization theorem of Beilinson-Bernstein is via the Geometric Langlands correspondence. We state this briefly here to provide the idea. The Langlands correspondence is a bijection between two sets of objects associated to a field  $F$  and a reductive connected algebraic group  $G$ . Originally it was formulated when  $F$  was either a number field or  $F = \mathbb{F}_q(X)$  the field of rational functions on a smooth projective curve  $X$  defined over a finite field  $\mathbb{F}_q$ . For simplicity we will restrict ourselves to the function field case  $F = \mathbb{F}(X)$  and a split connected simple algebraic group  $G$  defined over  $\mathbb{F}_q$ . For a closed point  $x \in X$  let  $\mathcal{O}_x$  be the completion of the local ring of  $x$ ; i.e.  $\mathcal{O}_x = \mathbb{F}_{q_x}[[t]]$  and  $K_x$  its field of fractions, where  $q_x = q^{\deg x}$ .

A representation  $\pi_x$  of the group  $G_x = G(K_x)$  on a  $\overline{\mathbb{Q}_p}$ -vector space is called smooth if the stabilizer of any vector is an open subgroup of  $G_x$ . A smooth irreducible representation is called unramified if there exists a non-zero vector  $v_x$  that is invariant with respect to the subgroup  $K_x = G(\mathcal{O}_x)$ . Such a vector is then unique up to multiplication in  $\overline{\mathbb{Q}_l}^\times$ .

There is a correspondence between the equivalence classes of irreducible unramified representations of  $G_x$  and conjugacy classes in the Langlands dual group  $G^L(\overline{\mathbb{Q}_p})$ . This correspondence can be described via the so called principal series representations of  $G_x$ . For instance if  $G_x = GL_n$  any semisimple conjugacy class in  $G^L(\overline{\mathbb{Q}_p})$  contains a diagonal matrix  $y = \text{diag}[y_1, \dots, y_n]$ ,  $y_i \in \overline{\mathbb{Q}_p}$  defined up to permutations. To such a class we can associate a character of the upper Borel subgroup by

$$(143) \quad \chi_y(b) = (q_x^{1-n} y_1)^{\nu_x(b_{11})} (q_x^{2-n} y_2)^{\nu_x(b_{22})} \dots y_n^{\nu_x(b_{nn})}$$

This character then defines an induced representation of  $GL_{n,x}$  on the space of locally constant functions  $f$  on  $GL_{n,x}$  such that  $f(bx) = \chi_y(b)f(g)$  for all  $b \in B$  and  $f \in GL_{n,x}$ . This is a principal series representation. It is known that it contains exactly one irreducible unramified component, which depends only on the conjugacy class of  $y$ .

*Thus one obtains a correspondence between the conjugacy classes in  $G^L(\overline{\mathbb{Q}_l})$  and  $G_x$ -modules which is called the local Langlands correspondence.*

Let  $\mathbb{A}$  be the ring of adels of  $F$ . Note that  $F$  diagonally embeds into  $\mathbb{A}$ . The group  $G(\mathbb{A})$  diagonally acts on  $C(G(F) \setminus G(\mathbb{A}))$  of locally constant functions on the quotient  $G(F) \setminus G(\mathbb{A})$ . An irreducible representation of  $G(\mathbb{A})$  is called automorphic if it appears in the decomposition of  $C(G(F) \setminus G(\mathbb{A}))$ . When such a representation is unramified the tensor product of  $K_x$ -invariant vectors  $\otimes_x v_x$  is right invariant with respect to the compact subgroup  $K = \otimes_x K_x$ . Hence it defines a function on the double coset  $G(F) \setminus G(\mathbb{A})/K$ , which is the set of isomorphism classes of  $G$ -bundles on  $X$ .

*The global Langlands correspondence states that; an irreducible unramified representation  $\otimes'_x \pi_x$  is automorphic iff there exists a continuous homomorphism  $\sigma : \pi_1(X) \rightarrow G^L(\overline{\mathbb{Q}_l})$  such that each  $\pi_x$  corresponds to the conjugacy class  $\sigma(\text{Fr}_x)$  in the case of local Langlands correspondence.*

Now we consider  $F = \mathbb{C}(X)$  where  $X$  is an algebraic curve and the loop group  $LG = G((t))$  in this case. Let  $G_{in} = G[[t]]$  and  $\mathcal{O}_{crit}^0$  be the category of unramified  $\widehat{\mathfrak{g}}$ -modules of critical level, which consist of modules on which the action of  $\mathfrak{g}_{in} = \mathfrak{g}[[t]]$  is locally finite and which contain  $\mathfrak{g}_{in}$ -invariant vector. On such modules the action of  $\mathfrak{g}_{in}$  can be integrated to an action of the Lie

group  $G_{in}$ . The analogue of a conjugacy class in the group  $G^L$  is a regular  $\mathfrak{g}^L$ -oper on the formal disc. The analogue of the local Langlands correspondence is the following:

*Each regular  $\mathfrak{g}^L$ -oper  $\rho_x$  on the formal disc defines an irreducible  $\widehat{\mathfrak{g}}$ -module of critical level.*

Wakimoto modules are analogues of representations of principal series. Suppose that we are given a  $\mathfrak{g}^L$ -oper  $\rho_x$  for each point  $x \in X$ . Let  $V^{\rho_x}$  be the corresponding  $\widehat{\mathfrak{g}}$ -module of critical level defined at the end of Session (4). Set  $\mathfrak{g}(\mathbb{A}) = \prod'_x \mathfrak{g}((t_x))$  and  $\widehat{\mathfrak{g}}(\mathbb{A})$  be its one dimensional central extension. The product  $\otimes'_x V^{\rho_x}$  is naturally a  $\widehat{\mathfrak{g}}(\mathbb{A})$ -module. In this case one can assign a twisted  $D$ -module to the  $\widehat{\mathfrak{g}}(\mathbb{A})$ -module  $\otimes_x M_x$  by the localization functor defined before;

$$(144) \quad loc : \otimes'_x V^{\rho_x} \longmapsto \Delta(\otimes_x M_x)$$

The action of  $\mathfrak{g}_{in}$  on  $M_x$  can be integrated to an action of  $G_{in}$ . Therefore the  $D$ -module is  $K$ -equivariant and descends to a  $D$ -module on  $\mathcal{M}$  in (79), denoted by  $\Delta(\otimes_x M_x)$ . Lets specialize to  $k = -h^\vee$ . The  $\widehat{\mathfrak{g}}(\mathbb{A})$ -module  $\otimes_x V^{\rho_x}$  is called weakly automorphic if  $\Delta(\otimes_x V^{\rho_x}) \neq 0$ . The weak version of the global Langlands correspondence over  $\mathbb{C}$  states as follows:

*The  $\widehat{\mathfrak{g}}(\mathbb{A})$ -module  $\otimes_x V^{\rho_x}$  is weakly automorphic iff there exists a globally defined regular  $\mathfrak{g}^L$ -oper  $\rho$  on  $X$  such that for each  $x$ , the  $\rho_x$  is the restriction of  $\rho$  to a small disc around  $x$ .*

In case of a finite field, the automorphic function on the double coset  $G(F) \backslash G(\mathbb{A})/K$  corresponding to an automorphic representation  $\otimes_x \pi_x$  is an eigenfunction of the Hecke operators. Its eigenvalues are given by traces of  $\sigma(Fr_x)$  on the finite dimensional representation of  $G^L(\overline{\mathbb{Q}_l})$ . The analogues of the Hecke operators over  $\mathbb{C}$  are certain operations on  $D$ -modules on  $M_G(X)$ . Using these one can state the global Langlands correspondence as:

*$\Delta_\rho$  is an eigensheaf with respect to these Hecke correspondences with eigenvalues given in terms of  $G^L$ -local system defined by  $\rho$ . More generally one expects that for any homomorphism  $\sigma : \pi_1(X) \rightarrow G^L(\mathbb{C})$  or equivalently a  $G^L$ -local system on  $X$ , or a flat  $G^L$ -bundle over  $X$ , there exists a unique  $D$ -module  $F_\sigma$  on  $M_G(X)$ , which is automorphic with respect to  $\sigma$ , in the sense it is an eigensheaf of Hecke correspondences with eigenvalues given in terms of  $\sigma$ , [F2].*

- (5) **Knizhnik-Zamolodchikov (KZ)-equations:** We keep this section as a well known example in conformal field theory. Assume  $\mathfrak{g}$  is a finite dimensional semisimple Lie algebra with invariant bilinear form  $\kappa$ . Let  $\widehat{\mathfrak{g}}_k$  be the affine lie algebra with level  $k$  and dual Coxeter number  $h^\vee$ . The null vector of a  $\widehat{\mathfrak{g}}_k$ -module defines differential equations

$$(145) \quad (k+2) \frac{\partial}{\partial z_i} \Psi = \sum_{i \neq j} \frac{\Omega_{ij}}{z_i - z_j} \Psi$$

called KZ-equations, where  $\Omega_{ij} = \sum_a J^a J_a$  are matrices.  $\rho_i$  are the representations associated to local monodromies.  $J^a$  and  $J_a$  are dual basis with respect to the invariant bilinear form  $\kappa$  on  $\mathfrak{g}$ . In case that  $\mathfrak{g}$  is semisimple, its local system corresponds to representations of the braid group

$$(146) \quad \theta : B_n \rightarrow V_1 \otimes \dots \otimes V_n$$

as the holonomy of the Hamiltonian system (145) (by Riemann-Hilbert correspondence or non-abelian-Hodge theorem). A more concrete way to write this equation is

$$(147) \quad dw = \sum_{1 \leq i < j \leq n} \frac{dz_i - dz_j}{z_i - z_j} A_{ij} w$$

which can be written as an equation of a flat connection  $\nabla^{KZ} = d - \Gamma$  where  $\Gamma = \sum (dz_i - dz_j) A_{ij} / (z_i - z_j)$ . In the simplest case of 3-correlations over  $\mathbb{P}^1 \setminus \{0, 1, \infty\}$  the equation (147) can be reduced to a one variable equation

$$(148) \quad \frac{d\phi}{dx} = \left( \frac{A}{x} - \frac{B}{1-x} \right) \phi, \quad \phi \in W\{x\}[\log x]$$

after suitable change of variables, where  $A, B \in \text{End}(W)$  for a  $\mathfrak{g}$ -module  $W$  are diagonal matrices. The system of solutions for (103) can be described in two ways which we explain as follows

- Suppose  $\{\lambda\}$  is the set of eigenvalues of  $A$  such that all eigenvalues of  $A$  are contained in  $\cup_\lambda \lambda + \mathbb{N}$ . Thus for each  $\lambda$  there is a set  $\{\lambda + N_j^\lambda\}_{j=0}^{J_\lambda}$  such that  $0 = N_0^\lambda < \dots < N_{J_\lambda}^\lambda$ . Denote by  $\pi_\lambda^A$  the projection onto the  $\lambda$ -eigenspace. Then a basic but perhaps not short calculation shows that for any  $w \in W$  there is a unique solution to (148) of the form

$$(149) \quad \phi_w^A(x) = \sum_\lambda \sum_j \sum_{i \geq N_j^\lambda} w_{i,j}^{(\lambda)} x^{\lambda+i} (\log x)^j$$

where  $w_{0,0}^{(\lambda)} = \pi_\lambda^A(w)$  and for each  $j > 0$  we have  $\pi_{\lambda_{N_j^\lambda,0}^A}^A(w_{\lambda_{N_j^\lambda,0}^{(\lambda)}}) = \pi_{\lambda_{N_j^\lambda,0}^A}^A(w)$ .

- With similar set up but this time looking at the  $B$ -eigenvalues one shows that; for any  $w \in W$  there is a unique solution of

$$(150) \quad \frac{d\phi}{dy} = \left( \frac{B}{y} - \frac{A}{1-y} \right) \phi, \quad \phi \in W\{y\}[\log y]$$

of the form

$$(151) \quad \phi_w^B(y) = \sum_{\mu} \sum_k \sum_{i \geq M_k^{\mu}} w_{i,k}^{(\mu)} y^{\mu+i} (\log y)^k$$

In the first case the map

$$(152) \quad \phi_A : w \mapsto \phi_w^A(z)$$

defines a  $\mathfrak{g}$ -isomorphism between  $W$  and the solution system of the KZ-equation. In the second case the map

$$(153) \quad \phi_B : w \mapsto \phi_w^B(1-z)$$

The  $\mathfrak{g}$ -automorphism  $\Phi_{KZ} = \phi_B^{-1} \phi_A$  of  $W$  is called the Drinfeld associator of  $W$ , which can be interpreted by the intertwining operators explaining the tensor structure on the solutions of KZ-equation, [M].

The system of differential equations (145) or (148) define a regular holonomic differential system whose solutions parametrize Hodge structures on the punctured plane. The procedure of defining variation of Hodge structure in the geometric setting is similar. Generally variation of Hodge structure arise from system of differential equations. The regularity condition implies that the resulting local system of solutions can be extended over the puncture. According to the Riemann-Hilbert correspondence the regular local system of Hodge structure correspond to regular holonomic  $D$ -modules, in our case over the punctured plane. A  $D$ -module by definition is a module over the ring of micro-differential operators on the base manifold. In special set up these modules also satisfy a compatible action of a Lie group  $G_+$ , which makes them a Harish-Chandra pair. In this sense the formalism of the previous sessions will apply. In other words the local system of solutions (149) and (151) are local system of Hodge structure with some  $\mathfrak{g}$ -action coming from the internal symmetries of the differential equation.

- (6) **Conformal blocks:** As was mentioned a vertex algebra is called conformal if among the vertex operators, there is the generating function of the basis element of the Virasoro algebra. Such algebras or their modules become a module over Virasoro algebra. To a conformal vertex algebra  $V$  one can



assign a vector bundle  $\mathcal{V}$  over an algebraic curve as a base manifold  $X$ . A vertex operator in this set up may be interpreted as a section of the dual bundle  $\mathcal{V}^*$  on the punctured disc  $D_x^\times$  with values in  $End(V_x)$ , which may be written as

$$(154) \quad A \otimes z^n dz \longmapsto Res_{z=0} Y(A, z) z^n dz$$

In the affine Kac-Moody case, given a  $\widehat{\mathfrak{g}}$ -module  $M_x$  we define its space of co-invariants as the quotient  $M_x / \mathfrak{g}_{out}(x).M_x$ . The space of conformal blocks is the dual space

$$(155) \quad (M_x / \mathfrak{g}_{out}(x).M_x)^\vee = Hom_{\mathfrak{g}_{out}(x)}(M_x, \mathbb{C})$$

The basis elements of  $\widehat{\mathfrak{g}}$  are given by  $J^a(z)$ , and  $J^a(z)dz$  is naturally a one form. Then  $\phi \in M_x^*$  is a conformal block if and only if  $\langle \phi, J^a(z).A \rangle$  has regular singularity at  $x$ .

On the moduli  $\widehat{\mathcal{M}}_g$  of pairs  $(X, z)$  where  $z$  is a coordinate frame on  $X$ , one can define a Harish-Chandra pair  $(Der(K_X), Aut(\mathcal{O}_X))$  by applying the formalism of Harish-Chandra localization to a conformal vertex algebra  $V$  to obtain twisted  $D$ -modules of a sheaf of co-invariants on  $\widehat{\mathcal{M}}_g$ . This construction can be generalized to other moduli spaces such as moduli of  $G$ -bundles on  $X$  etc.... If we are given a vertex algebra  $V$  and a Harish-Chandra pair  $(\mathcal{B}, \mathcal{G}_+)$  of what is called internal symmetries (in case of Hodge structure come from Mumford-Tate group), then to any such data one can attach a twisted space of co-invariants  $H_\tau(X, z)$ . Here the Lie algebra action of  $\mathcal{B}$  is induced by the Fourier coefficients of vertex operators in  $V$ . And  $\mathcal{G}_+$  consists of certain symmetries of a geometric data  $\mathcal{H}$  (which in our case is the local system of Hodge structure) on the punctured disc  $D^\times$ . As  $\tau$  varies  $H_\tau(X, z)$  combine into a twisted  $D$ -module  $\Delta(V)$  on the moduli  $\mathcal{M}_\Phi$  of the data  $\Phi$  on  $X$ , [FB].

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